

# Matrix integrals and generating functions for permutations and one-face rooted hypermaps

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## Abstract

Closed-form generating functions for counting one-face rooted hypermaps with a known number of darts by number of vertices and edges is found, using matrix integral expressions relating to the reduced density operator of a bipartite quantum system. A recursion relation for these generating functions is also found. The method for computing similar generating functions for two-face rooted hypermaps by number of vertices and edges is outlined.

**Keywords:** enumeration, rooted hypermap, bipartite quantum system, matrix integral, generating function

## 1 Introduction

The link between enumerative combinatorics and matrix integration has been discussed for some time [21, 10]. In the physics community it has seen extensive use in the approximation of quantities in quantum field theory, with the problem of evaluating matrix integrals being replaced by that of evaluating summations over graphs and other combinatoric objects [3, 9]. A familiar related concept is that of Feynman diagrams, where a path integral is expressed as a sum of terms expressed using graphs with particular properties, allowing the path integral to be approximated as a sum over a finite number of graphs.

It can be used in the opposite sense as well; just as counting graphs may be used to evaluate intractable integrals, in some cases it is possible to evaluate the matrix integrals exactly in order to find generating functions which can be used to count graphs. This has gained relatively little attention, although examples are known, such as counting one-face maps with given genus and number of edges [2, 7, 21].

However, the fact that matrix integration is potentially able to produce exact generating functions for combinatoric classes makes this a very powerful tool; generating functions have many uses in enumerative combinatorics, as basic operations on generating functions such as addition and multiplication have combinatorial meanings in terms of unions and products of sets [17, pp 3-4].

In this paper we further demonstrate the power of matrix integrals by deriving a closed-form generating function for counting rooted hypermaps with one face and  $r$  darts (which are defined in Section 2.3) by number of edges and vertices. This is done using matrix integral expressions arising in the study of finite-dimensional bipartite quantum systems (which are defined in Section 2.1)

Considerable work has been done on the enumeration of hypermaps prior to this. Explicit expressions for the number of hypermaps with a given number of darts have been found for genus zero (planar) [19] and genus one (toroidal) [1] hypermaps, and recursive formulae have then been used to extend this to higher genus [12]. Other methods have also been used, including iterating explicitly through all hypermaps of a given size and computing their properties; this has in particular been used to count rooted hypermaps for any number of vertices, edges, faces and up to twelve darts [18].

In comparison to this past work, the method presented here is neither limited to a single genus, nor reliant on recursion (the generating function for hypermaps with  $r$  darts is a closed-form function of  $r$  and not dependent on the generating functions for lower  $r$ ) or direct enumeration. It is also, as far as we are aware, the first time that matrix integrals have been used to derive generating functions for hypermaps.

Having said this, we will also show that there is a recursive solution to this problem, which can find the generating function for  $r$  darts in terms of those for  $r - 1$  and  $r - 2$  darts using a simple recursion relation (see Section 3.2). The recursive method can compute generating functions even more efficiently if the aim is to find all such functions up to a particular order, although the ability to compute a single generating function without computing the preceding ones is lost.

The one-face rooted hypermap case is in fact a demonstration of a much more general property of certain matrix integrals (specifically polynomial integrals over the unit sphere), relating them to sums over the symmetric group of permutations. We will demonstrate the broader link using the simpler case of enumeration of permutations by number of cycles as an example, again deriving the result in the form of a generating function (see Section 2.2). We will also show our method's broader applicability to rooted hypermaps in particular by discussing how it may be used to enumerate rooted hypermaps with two faces as well (again by number of darts, edges and vertices; see Section 3.3).

## 2 Quantum systems and hypermaps

### 2.1 Bipartite quantum systems

See [16, pp 180-186] for an in-depth introduction to density operators.

A *finite-dimensional bipartite quantum system*  $S$  is a system with an associated Hilbert space  $\mathcal{H} \equiv \mathbb{C}^{mn}$  for positive integers  $m$  and  $n$ , such that it can be decomposed into the union of two systems  $A$  and  $B$  with Hilbert spaces  $\mathcal{H}_A \equiv \mathbb{C}^m$  and  $\mathcal{H}_B \equiv \mathbb{C}^n$  respectively, such that  $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ . A state of this system is described by a *density operator*  $\hat{\rho}$ , which is a positive Hermitian operator with unit trace acting on  $\mathcal{H}$ . We will only be considering *pure states* of  $S$ , which are defined as those states where  $\hat{\rho}$  is a projection operator.

In this paper we are interested in the *reduced density operator* for the subsystem  $A$ , denoted  $\hat{\rho}^A$ , which is found by taking the partial trace of  $\hat{\rho}$  over  $\mathcal{H}_B$ .  $\hat{\rho}^A$  is not necessarily in a pure state even when  $\hat{\rho}$  is pure, meaning that  $\text{Tr}[(\hat{\rho}^A)^r] \neq 1$ . We wish to average this expression over all possible pure states of  $S$ , and will do so by parametrising  $\hat{\rho}$  in terms of a unit vector  $|z\rangle$  in  $\mathcal{H}$  (i.e. such that  $\hat{\rho} = |z\rangle\langle z|$ ) and integrating with respect to the invariant measure on the unit sphere  $S^{2mn-1}$  containing  $|z\rangle$  (this measure has been used for this purpose in other papers, such as [11, 13] – in [11] it is referred to as the “Haar measure”).

We express the mean  $\langle \text{Tr}[(\hat{\rho}^A)^r] \rangle$  in terms of the function  $P_r(m, n)$ , defined in Theorem 1; this function is important as we will prove in Section 2.3 that it is equivalent to the generating function for enumerating one-face rooted hypermaps with  $r$  darts, as we will prove in Theorem 3 in Section 2.3.

**Theorem 1.** *If  $\hat{\rho}^A$  is the reduced density operator of an  $m$ -dimensional subsystem of an  $mn$ -dimensional bipartite quantum system  $S$ , then for any integer  $r$ ,*

$$\langle \text{Tr}[\hat{\rho}_A^r] \rangle = \frac{\Gamma(mn)}{\Gamma(mn+r)} P_r(m, n). \quad (1)$$

Here  $P_r(m, n)$  is defined as

$$P_r(m, n) = \frac{\partial}{\partial \alpha_{a_1 b_1}} \frac{\partial}{\partial \alpha_{a_2 b_2}} \cdots \frac{\partial}{\partial \alpha_{a_r b_r}} (\alpha_{a_1 b_2} \cdots \alpha_{a_{r-1} b_r} \alpha_{a_r b_1}) \Big|_{\alpha=0}, \quad (2)$$

where  $\alpha_{ab}$  are the components of an  $m \times n$  real matrix  $\alpha$ .

*Proof.* Let  $\{|\phi_1^A\rangle, \dots, |\phi_m^A\rangle\}$  and  $\{|\phi_1^B\rangle, \dots, |\phi_n^B\rangle\}$  be two orthonormal basis sets which span  $\mathcal{H}_A$  and  $\mathcal{H}_B$  respectively. In terms of this basis we parametrise a general vector  $|z\rangle \in \mathcal{H}$  as

$$|z\rangle = z_{ab} |\phi_a^A\rangle \otimes |\phi_b^B\rangle,$$

where  $z$  is an  $m \times n$  complex matrix, and the convention of summing over repeated indices is assumed; in this and all future working, any index represented

by the letter  $a$  runs from 1 to  $m$  and any index represented by the letter  $b$  runs from 1 to  $n$ . In terms of the components of  $z$ , the inner product of two such vectors is

$$\langle x|z\rangle = x_{ab}^* z_{ab},$$

so the condition for  $|z\rangle$  to be a unit vector is

$$\langle z|z\rangle = z_{ab}^* z_{ab} = 1.$$

We now express  $\hat{\rho}^A$  using the *reduced density matrix*  $\rho^A$ , which in our chosen basis has components

$$\rho_{a_1 a_2}^A = \langle \phi_{a_1}^A | \hat{\rho}^A | \phi_{a_2}^A \rangle = z_{a_1 b} z_{a_2 b}^*,$$

and use these components to construct the following matrix-integral expansion for  $\langle \text{Tr}[(\hat{\rho}^A)^r] \rangle$ :

$$\langle \text{Tr}[(\hat{\rho}^A)^r] \rangle = \frac{1}{S_{2mn-1}} \int_{S^{2mn-1}} d\sigma z_{a_1 b_1} z_{a_2 b_1}^* z_{a_2 b_2} z_{a_3 b_2}^* \cdots z_{a_r b_r} z_{a_1 b_r}^*, \quad (3)$$

where  $d\sigma$  is the invariant volume element on the hypersphere  $S^{2mn-1}$  and

$$S_{2mn-1} = \frac{2\pi^{mn}}{\Gamma(mn)}$$

is the volume of  $S^{2mn-1}$ .

(3) is a polynomial in the components of  $z$  and  $z^*$  containing only terms of order  $2r$ . It is known that such an integral performed over the unit sphere can be converted into a Gaussian integral [6]. Following this procedure, we multiply (3) by the expression

$$\frac{2}{\Gamma(mn+r)} \int_0^\infty \lambda^{2mn+2r-1} e^{-\lambda^2} d\lambda,$$

which equals unity by the definition of the gamma function. Then we define  $x = \lambda z$ , and note that  $x_{ab}^* x_{ab} = \lambda^2$  and  $d^{2mn}x = \lambda^{2mn-1} d\lambda d\sigma$ . Using this change of variables, we get that

$$\langle \text{Tr}[(\hat{\rho}^A)^r] \rangle = \frac{\Gamma(mn)\pi^{-mn}}{\Gamma(mn+r)} \int_{\mathbb{C}^{mn}} d^{2mn}x e^{-x_{ab}^* x_{ab}} x_{a_1 b_1} x_{a_2 b_1}^* \cdots x_{a_r b_r} x_{a_1 b_r}^*. \quad (4)$$

Next we define a pair of real  $m \times n$  matrices  $\alpha$  and  $\beta$ . We use these to remove the polynomial part from (4):

$$\begin{aligned} \langle \text{Tr}[(\hat{\rho}^A)^r] \rangle &= \frac{\Gamma(mn)\pi^{-mn}}{\Gamma(mn+r)} \frac{\partial}{\partial \alpha_{a_1 b_1}} \frac{\partial}{\partial \beta_{a_2 b_1}} \cdots \frac{\partial}{\partial \alpha_{a_r b_r}} \frac{\partial}{\partial \beta_{a_1 b_r}} \\ &\quad \int_{\mathbb{C}^{mn}} d^{2mn}x e^{-x_{ab}^* x_{ab} + \alpha_{ab} x_{ab} + x_{ab}^* \beta_{ab}} \Big|_{\alpha, \beta=0} \\ &= \frac{\Gamma(mn)}{\Gamma(mn+r)} \frac{\partial}{\partial \alpha_{a_1 b_1}} \frac{\partial}{\partial \beta_{a_2 b_1}} \cdots \frac{\partial}{\partial \alpha_{a_r b_r}} \frac{\partial}{\partial \beta_{a_1 b_r}} e^{\alpha_{ab} \beta_{ab}} \Big|_{\alpha, \beta=0}. \end{aligned}$$

As the derivatives all commute with each other, we can choose what order to perform them in. Choosing to perform the  $\beta$ -derivatives first, we get

$$\begin{aligned}\langle \text{Tr}[(\hat{\rho}^A)^r] \rangle &= \frac{\Gamma(mn)}{\Gamma(mn+r)} \frac{\partial}{\partial \alpha_{a_1 b_1}} \cdots \frac{\partial}{\partial \alpha_{a_r b_r}} (\alpha_{a_1 b_1} \cdots \alpha_{a_r b_r}) \Big|_{\alpha=0} \\ &\equiv \frac{\Gamma(mn)}{\Gamma(mn+r)} P_r(m, n).\end{aligned}$$

□

## 2.2 Permutations

In Theorem 1 we have shown that the matrix integral (3) (which has the form of a polynomial function of matrix components integrated over the unit sphere) can equivalently be expressed as the multiderivative expression (2). Evaluation of  $P_r(m, n)$  based on (2) is directly linked to the symmetric group of permutations in that it involves summing over all possible pairings of  $\frac{\partial}{\partial \alpha}$  terms with  $\alpha$  terms; there are  $r!$  such pairings, corresponding to the different permutations on  $r$  elements. In this section we will demonstrate this fact more clearly by using a related matrix integral to enumerate the set of permutations on  $r$  elements by number of cycles.

Before we begin, it should be noted that the result in this case is already known; the number of permutations on the set  $\{1 \dots r\}$  with  $k$  cycles is the unsigned Stirling number of the first kind  $\mathfrak{s}(r, k)$  [4, p 234], and these have a generating function [4, p 213]

$$\sum_{k=1}^r \mathfrak{s}(r, k) m^k = \frac{\Gamma(m+r)}{\Gamma(m)}. \quad (5)$$

Our aim here is to show how the matrix integrals are used to find generating functions, however, so we will in this section derive the generating function (5) directly without considering the properties of the counting functions themselves (denoted  $c_r(k)$  from now on to make it clear that nothing of their value is being assumed).

**Theorem 2.** *Let  $c_r(k)$  be the number of permutations on the set  $R = \{1 \dots r\}$  with a decomposition into  $k$  disjoint cycles. Then*

$$\sum_{k=1}^r c_r(k) m^k = \frac{\Gamma(m+r)}{\Gamma(m)}$$

for any  $r \geq 1$ .

*Proof.* Define

$$P_r(m) = \frac{2}{\Gamma(m)} \int_0^\infty x^{2m+2r-1} e^{-x^2} dx. \quad (6)$$

Using the definition of the gamma function, we know that

$$P_r(m) = \frac{\Gamma(m+r)}{\Gamma(m)}. \quad (7)$$

Let  $u$  be a random vector in  $\mathbb{C}^m$ , where  $x = \sqrt{u \cdot u}$ . Noting that  $d^{2m}u = x^{2m-1}dx d\sigma$  (where  $d\sigma$  is the invariant volume element on the sphere  $S^{2m-1}$ ), we express (6) as

$$\begin{aligned} P_r(m) &= \frac{2}{\Gamma(m)} \int_0^\infty (u \cdot u)^r e^{-u \cdot u} x^{2m-1} dx \\ &= \frac{2}{\Gamma(m)} \frac{\int_{\mathbb{C}^m} (u \cdot u)^r e^{-u \cdot u} d^{2m}u}{\int_{S^{2m-1}} d\sigma}. \end{aligned}$$

$\int_{S^{2m-1}} d\sigma = 2\pi^m / \Gamma(m)$  is the volume of the sphere  $S^{2m-1}$ , so

$$P_r(m) = \pi^{-m} \int_{\mathbb{C}^m} (u \cdot u)^r e^{-u \cdot u} d^{2m}u.$$

We then evaluate the integral by defining two real-valued  $m$ -vectors  $\alpha$  and  $\beta$ , and writing

$$\begin{aligned} P_r(m) &= \pi^{-m} \left( \frac{\partial}{\partial \alpha_i} \frac{\partial}{\partial \beta_i} \right)^r \int_{\mathbb{C}^m} e^{-u \cdot u + \alpha \cdot u + u \cdot \beta} d^{2m}u \Big|_{\alpha, \beta=0} \\ &= \left( \frac{\partial}{\partial \alpha_i} \frac{\partial}{\partial \beta_i} \right)^r e^{\alpha \cdot \beta} \Big|_{\alpha, \beta=0}, \end{aligned}$$

where the convention of summation over the repeated index  $i$  (running from 1 to  $m$ ) is assumed. The derivatives all commute, so we can choose to perform all of the  $\beta$ -derivatives first. After doing so we get

$$P_r(m) = \frac{\partial}{\partial \alpha_{i_1}} \dots \frac{\partial}{\partial \alpha_{i_r}} (\alpha_{i_1} \dots \alpha_{i_r}) \Big|_{\alpha=0}. \quad (8)$$

To fully expand the derivatives in this expression out, we need to account for every possible pairing of the  $r$  derivative terms with the  $r$   $\alpha_i$  terms, leading to a summation over all permutations  $\sigma$  on  $R$  (i.e. all permutations in the symmetric group  $Sym_r$ ). Therefore,

$$P_r(m) = \sum_{\sigma \in Sym_r} \prod_{s=1}^r \frac{\partial}{\partial \alpha_{i_s}} \alpha_{i_{\sigma(s)}} = \sum_{\sigma \in Sym_r} \prod_{s=1}^r \delta[i_s, i_{\sigma(s)}]$$

where  $\delta[\dots]$  is the Kronecker delta.

In the term for a given  $\sigma$  every  $i$  index in the product is implicitly contracted over, and any cycle in  $\sigma$  gives rise to a single self-contained contraction. e.g. a cycle (134) would result in a term looking like

$$\delta[i_1, i_3] \delta[i_3, i_4] \delta[i_4, i_1] = \delta[i_1, i_1] = m.$$

Each cycle therefore contributes a factor of  $m$ , such that the term corresponding to  $\sigma$  is equal to  $m^a$ , where  $a$  is the number of cycles in  $\sigma$ . This means that

$$P_r(m) = \sum_{k=1}^r c_r(k) m^k$$

where  $c_r(k)$  is the number of permutations on  $R$  with  $k$  cycles. (7) therefore gives

$$\sum_{k=1}^r c_r(k) m^k = \frac{\Gamma(m+r)}{\Gamma(m)}.$$

□

This proof demonstrates that the procedure for evaluating mutiderivative expressions such as (8) involves a summation over permutations, and that the resulting function acts as a generating function counting the permutations by number of cycles. In the next section we will explain how rooted hypermaps can be expressed using permutations, and then evaluate (2) using the above method to show that it is the generating function for rooted one-face hypermaps with  $r$  darts.

This generating function is also directly related to  $P_r(m, n)$  through the identity

$$P_r(m) = P_r(m, 1).$$

We will state the reason for this at the end of the next section.

## 2.3 Hypermaps

The following definitions come from [10].

A *hypergraph*  $G$  is a pair  $\{V, E\}$  consisting of a (non-empty) set  $V$  of vertices and a family  $E$  of edges, each edge being a non-empty family of vertices from  $V$ . This is a generalisation of the standard notion of a graph; whereas in graphs each edge must contain exactly two vertices, in hypergraphs they can contain any positive number of vertices. An edge in a hypermap can contain any given vertex more than once, and each edge-vertex connection is called a *dart*.

A *map* is an embedding of a graph onto an orientable surface such that the graph's complement consists of regions (called faces) which are isomorphic to the open unit disc; such an embedding is called a *2-cell embedding*. Through this embedding, a map gains a *genus* which is equal to the genus of the surface it's embedded in. There is a bijection between hypergraphs and bicoloured bipartite graphs [19], so the concept of maps generalises naturally to hypergraphs, such that a *hypermap* can be defined as an 2-cell embedding of a hypergraph on an orientable surface. The definitions of faces and genus carry over.

In addition, a *rooted hypermap* is a hypermap where one dart has been labelled as a "root", i.e. an isomorphism between rooted hypermaps must preserve the identity of the root as well as the hypermap's connectivity.

There is another definition of a hypermap (also from [10]) which will be of more use here. It is equivalent to the above, but defines the embedding in a combinatorial manner without having to consider actual surfaces. This defines hypermaps in terms of *3-constellations* - ordered triples  $\{\xi, \chi, \eta\}$  of permutations on  $r$  elements where (a) the group generated by  $\{\xi, \chi, \eta\}$  acts transitively on the set  $\{1, \dots, r\}$  and (b) the product  $\xi\chi\eta$  equals the identity.

**Definition 1.** A hypermap  $H$  with  $r$  darts is a 3-constellation  $\{\xi, \chi, \eta\}$  of permutations acting on the set  $\{1, \dots, r\}$ , which represents the hypermap's darts. The three permutations correspond to cycling the darts adjacent to any given face, edge or vertex<sup>1</sup> respectively in an anticlockwise direction around their adjacent object.

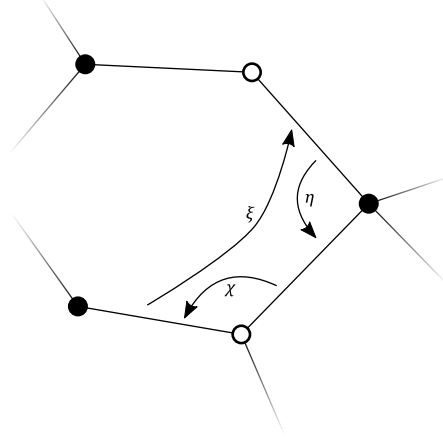


Figure 1: Dart mappings within a hypermap with 3-constellation  $\{\xi, \chi, \eta\}$ . The permutation  $\eta$  rotates each dart one place anti-clockwise around a vertex (black circle).  $\chi$  does the same around an edge (white circle).  $\xi$  rotates the dart anticlockwise around a face to the next dart adjacent to that face. The three mappings arranged like this show why  $\xi\chi\eta = 1$ .

Figure 1 shows how the three permutations act on the darts in a hypermap. It is clear that if e.g.  $\eta$  is applied a number of times equalling the order of a given vertex  $V$ , then all darts adjacent to  $V$  will map to themselves. Each vertex in a hypermap is therefore associated with a cycle in the permutation  $\eta$ . Similarly, the cycles in  $\xi$  and  $\chi$  are associated with faces and edges respectively.

<sup>1</sup>In the initial description of hypermaps given previously, which edge and vertex each dart is ‘adjacent to’ is clear, as darts were defined as connections between edges and vertices. If a hypermap is embedded on an orientable surface, you can uniquely define the “adjacent face” for a dart by moving anticlockwise from the dart around its adjacent vertex. While each dart borders two faces, it is only adjacent to one in this sense. Every other dart bordering a given face is adjacent to it.



We have already shown in Section 2.2 that matrix integration provides a way to enumerate permutations in terms of the number of cycles, so the correspondence between permutation cycles and edges/vertices in hypermaps now allows us to enumerate one-face rooted hypermaps by number of edges and vertices. This will make use of the quantum matrix integrals encountered in Section 2.1.

**Theorem 3.** *If  $\hat{\rho}^A$  is the reduced density operator of an  $m$ -dimensional subsystem of an  $mn$ -dimensional bipartite quantum system  $S$ , then for any integer  $r$*

$$\langle \text{Tr}[(\hat{\rho}^A)^r] \rangle = \frac{\Gamma(mn)}{\Gamma(mn+r)} \sum_{e,v} h_r^{(1)}(e,v) m^e n^v,$$

where  $h_r^{(1)}(e,v)$  is the number of rooted hypermaps with one face,  $r$  darts,  $v$  vertices and  $e$  edges, and the summation is over all possible values of these parameters for given  $r$ .

*Proof.* Using Theorem 1, we restate this as

$$P_r(m,n) = \sum_{e,v} h_r^{(1)}(e,v) m^e n^v.$$

We will prove this relation by considering how to evaluate  $P_r$  as a polynomial in  $m$  and  $n$ .

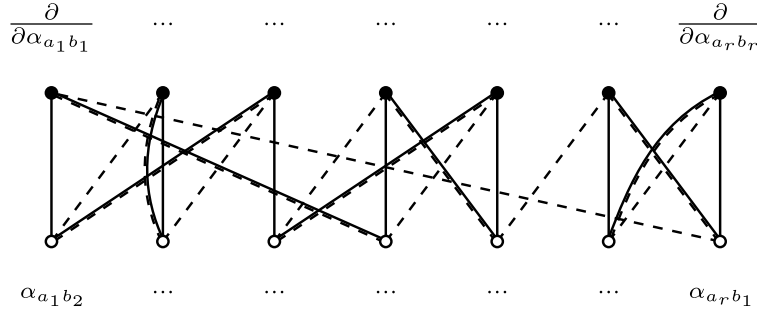


Figure 2: A diagram showing all contractions involved in evaluating a term in  $P_r(m,n)$  corresponding to a random permutation  $\sigma$  (in this case  $r = 7$  and the permutation  $\sigma$  this diagram corresponds to is  $(1453)(2)(67)$ ). The vertices correspond to the  $\frac{\partial}{\partial \alpha}$  and  $\alpha$  terms as labelled. The single solid and dotted lines represent contractions caused by shared  $a$  and  $b$  indices respectively. The double solid/dotted lines are the contractions caused by the pairings of  $\frac{\partial}{\partial \alpha}$  and  $\alpha$  terms, as given by  $\sigma$ . The term corresponding to this diagram is  $m^3 n^3$ , the exponents of  $m$  and  $n$  found by counting closed loops of solid and dotted lines respectively.

As with  $P_r(m)$  in 2.2,  $P_r(m,n)$  as given in (2) is a multiderivative function which can be expanded as a sum over permutations, each permutation

corresponding to different pairing of  $\frac{\partial}{\partial \alpha}$  terms and  $\alpha$  terms. Defining the cycle permutation

$$\xi = (123 \dots r), \quad (9)$$

we expand  $P_r$  as

$$P_r(m, n) = \sum_{\sigma \in \text{Sym}_r} \prod_{i=1}^r \frac{\partial}{\partial \alpha_{a_i b_i}} \alpha_{a_{\sigma(i)} b_{\xi \sigma(i)}} = \sum_{\sigma \in \text{Sym}_r} \prod_{i=1}^r \delta[a_i, \sigma(a_i)] \delta[b_i, \xi \sigma(b_i)], \quad (10)$$

where  $\delta[\dots]$  represents the Kronecker delta. The term corresponding to a given  $\sigma$  therefore consists of a completely contracted product of Kronecker deltas, some with dimension  $m$  (those with  $a$  indices) and some with dimension  $n$  (those with  $b$  indices). When these are contracted, they produce terms of the form  $m^A n^B$  where  $A$  is the number of cycles in  $\sigma$  and  $B$  the number of cycles in  $\xi \sigma$  (each cycle creates a completely closed loop of contractions such as  $\delta[a_1, a_2] \delta[a_2, a_4] \delta[a_4, a_1] = m$ ). See Figure 2 for a graphical representation of these cycles.

Therefore, the term  $m^A n^B$  associated with  $\sigma$  corresponds to a hypermap  $H_\sigma$  with 3-constellation  $\{\xi, \chi, \eta\} = \{\xi, \sigma, (\xi \sigma)^{-1}\}$ .  $A$  is the number of cycles in  $\chi$ , which is the number of edges in the hypermap, and  $B$  is the number of cycles in  $\eta$ , or the number of vertices in the hypermap. As  $\xi$  has only one cycle, these hypermaps necessarily have exactly one face.

This correspondence between permutations  $\sigma$  and hypermaps is not bijective, as two hypermaps with 3-constellations  $\{\xi, \chi, \eta\}$  and  $\{\xi', \chi', \eta'\}$  related by

$$\xi = \tau \xi' \tau^{-1}, \chi = \tau \chi' \tau^{-1}, \eta = \tau \eta' \tau^{-1} \quad (11)$$

for some permutation  $\tau$  are isomorphic to each other (the transformation (11) amounts to a simple reordering of the darts in the hypermap without changing its connectivity). Given that we have specified  $\xi$  exactly in (9), however, the only isomorphisms which preserve  $\xi$  are those where  $\tau$  is an integer power of  $\xi$ . To make the correspondence bijective we simply need to remove this cyclic equivalence of darts, and we do so by labelling one dart in the hypermap as the “root” to make it distinct from the others. The summation over permutations in (10) is therefore equivalent to a summation over all nonisomorphic rooted hypermaps.

This finally leads us to the conclusion that

$$P_r(m, n) = \sum_{e, v} h_r^{(1)}(e, v) m^e n^v,$$

and

$$\langle \text{Tr}[(\hat{\rho}^A)^r] \rangle = \frac{\Gamma(mn)}{\Gamma(mn + r)} \sum_{e, v} h_r^{(1)}(e, v) m^e n^v.$$

□

This proof quickly leads to the additional result:

**Corollary 1.** *There are  $r!$  rooted hypermaps with one face and  $r$  darts.*

*Proof.* This follows from the bijection proved in Theorem 3 between rooted hypermaps with one face and  $r$  darts and permutations in  $Sym_r$ . There are  $r!$  permutations in  $Sym_r$ , so there are  $r!$  such hypermaps.  $\square$

The bijection also gives a direct link back to the permutation generating function  $P_r(m)$  given in Section 2.2. As each one-face rooted hypermap corresponds bijectively to the permutation  $\chi$  in its 3-constellation representation, and the number of cycles in  $\chi$  gives the number of edges in the hypermap, then summing over all one-face rooted hypermaps with  $r$  darts and  $k$  edges gives the number of permutations on  $r$  elements with  $k$  cycles. We can equivalently express this as

$$P_r(m) = P_r(m, 1). \quad (12)$$

### 3 Computing generating functions

Theorem 3 immediately suggests a procedure for calculating  $P_r(m, n)$  i.e. by performing the summation over permutations given in (10). This can be done in approximately  $\mathcal{O}(r \cdot r!)$  time, as there are  $r!$  permutations to sum over and it would take  $\mathcal{O}(r)$  time to count the number of cycles for each one. However, this algorithm is equivalent to just enumerating the individual hypermaps, and it would be preferable to instead find a way of expressing the generating functions in a closed form as functions of  $m$ ,  $n$  and  $r$ .

Such a method does exist, and it arises from the fact that

$$P_r(m, n) = \frac{\Gamma(mn + r)}{\Gamma(mn)} \langle \text{Tr}[(\hat{\rho}^A)^r] \rangle.$$

While Theorem 1 is proven by evaluating this mean as a Gaussian integral, other methods for integrating such expressions exist. We used one such method in a recent paper, and derived the closed-form expression [5]

$$\langle \text{Tr}[(\hat{\rho}^A)^r] \rangle = \frac{\Gamma(mn)}{r\Gamma(mn + r)} \sum_{k=0}^{m-1} \frac{(-1)^k \Gamma(m + r - k) \Gamma(n + r - k)}{k! \Gamma(r - k) \Gamma(m - k) \Gamma(n - k)} \quad (13)$$

for integers  $m$  and  $n$  and real  $r$ . We can then rearrange this expression into a closed-form generating function as follows:

**Theorem 4.**

$$P_r(m, n) = \frac{1}{r!} \sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k} \frac{\Gamma(m+r-k)}{\Gamma(m-k)} \frac{\Gamma(n+r-k)}{\Gamma(n-k)} \quad (14)$$

for any positive integers  $r$ ,  $m$  and  $n$ .

*Proof.* From (1) and (13), we get that

$$P_r(m, n) = \frac{1}{r} \sum_{k=0}^{m-1} \frac{(-1)^k \Gamma(m+r-k) \Gamma(n+r-k)}{k! \Gamma(r-k) \Gamma(m-k) \Gamma(n-k)}$$

for integers  $m$  and  $n$  and real  $r$ . As we are only concerned with integers  $r$  in this paper, however, we can simplify it to

$$\begin{aligned} P_r(m, n) &= \frac{1}{r} \sum_{k=0}^{m-1} \frac{(-1)^k \Gamma(m+r-k) \Gamma(n+r-k)}{k! (r-k-1)! \Gamma(m-k) \Gamma(n-k)} \\ &= \frac{1}{r!} \sum_{k=0}^{m-1} (-1)^k \binom{r-1}{k} \frac{\Gamma(m+r-k)}{\Gamma(m-k)} \frac{\Gamma(n+r-k)}{\Gamma(n-k)}. \end{aligned}$$

The term inside the summation is zero if  $k > m$  or  $k > r$ , so we are free to change the upper bound of the summation to any integer greater than or equal to  $\lceil \min(m, r) \rceil$ . Therefore,

$$P_r(m, n) = \frac{1}{r!} \sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k} \frac{\Gamma(m+r-k)}{\Gamma(m-k)} \frac{\Gamma(n+r-k)}{\Gamma(n-k)}.$$

□

This expression is, for known  $r$ , a sum over a fixed finite number of polynomials in  $m$  and  $n$ , making it suitable for use as a generating function. It is also clearly symmetric in  $m$  and  $n$ , which reflects a fundamental property of rooted hypermaps – that there is a bijection from the set of rooted hypermaps to itself which consists of replacing each edge with a vertex and vice versa.

### 3.1 Computation

In order to compare this generating function with existing results, we calculated the coefficients of  $P_r$  computationally using two different methods:

1. *Enumerative* – enumerating permutations using Heap’s algorithm [8] and counting cycles using the scheme shown in figure 2.
2. *Generating-function* – expanding the generating function (14) explicitly.

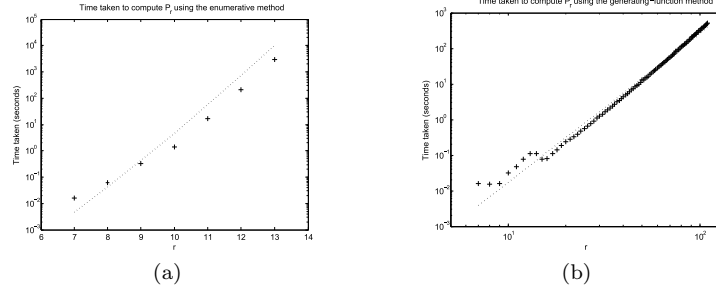


Figure 3: Graphs of time taken to compute  $P_r(m, n)$  for various values of  $r$  using (a) the enumerative method and (b) the generating-function method. The fitted curves are (a) the predicted  $\mathcal{O}(r \cdot r!)$  time dependence and (b) a power law fit with exponent 4.167. Both methods were computed on a 2012 Dell XPS 12 with a 2.49GHz processor, and cases for small  $r$  where the total run-time was less than the 16ms resolution of the computer’s clock have been omitted.

The enumerative method has a predicted complexity of  $\mathcal{O}(r \cdot r!)$ , which is reflected in the measured run-time of our C++ implementation (see figure 3a), in which all cases up to and including  $P_{13}$  (a total of 6 749 977 113 hypermaps) were computed in just under an hour. This can be compared to previous enumerative work by Walsh, in which he was able to compute 285 764 591 114 hypermaps (rooted, sensed and unsensed hypermaps with any number of faces and up to 12 darts) in a week [18]. This indicates that our algorithm was able to process hypermaps at approximately four times the rate that Walsh could; whether this improvement is due to the algorithm or hardware is unclear.

We also implemented the generating-function method in C++, using a specially designed class object to implement storage and manipulation of integer polynomials in two variables, along with the open-source MAPM library [15] to handle the arbitrarily large integers involved without loss of precision (calculating  $P_r(m, n)$  requires handling integers significantly larger than  $r!$ ). This method appears to run in polynomial time (based on measured run-times – see figure 3b), a significant improvement in terms of speed over the enumerative method. Indeed, the generating-function method reached  $P_{13}$  (as far as we took the enumerative method) in less than a second, and in the hour it took the enumerative method to get that far the generating-function method had reached  $P_{90}$ .

It must also be noted that the results given by these two methods agreed with each other in all cases covered by both, and also agreed with the values previously computed by Walsh [18].

### 3.2 Recursion relation

We now have an explicit closed-form expression for our generating functions  $P_r(m, n)$ , allowing us to compute any of them in isolation for arbitrary  $r$ . However, we can go further and also derive a recursion relation, allowing  $P_r(m, n)$  to be computed quickly in terms of the previous two generating functions. This arises from the fact that  $P_r(m, n)$  has the form of a hypergeometric sum, allowing us to use Zeilberger's algorithm for deriving recursion relations [14].

**Theorem 5.** *For any  $r \geq 1$ ,*

$$(r+3)P_{r+2}(m, n) = (2r+3)(m+n)P_{r+1}(m, n) + r[(r+1)^2 - (m-n)^2]P_r(m, n). \quad (15)$$

*Proof.* First, we write

$$P_r(m, n) = \sum_k F(r, k)$$

where

$$F(r, k) = \frac{(-1)^k}{r!} \binom{r-1}{k} \frac{\Gamma(m+r-k)}{\Gamma(m-k)} \frac{\Gamma(n+r-k)}{\Gamma(n-k)}$$

and the sum is over all integers  $k$  ( $F$  is zero when  $k < 0$  or  $k \geq r$ , so this is equivalent to (14)). The parameters  $m$  and  $n$  have been omitted for simplicity. Using the EKHAD Maple package [20], we find that this satisfies

$$\begin{aligned} (r+3)F(r+2, k) - (2r+3)(m+n)F(r+1, k) + r[(m-n)^2 - (r+1)^2]F(r, k) \\ = G(r, k+1) - G(r, k) \end{aligned} \quad (16)$$

where

$$\begin{aligned} G(r, k) = & \frac{(-1)^k}{(r+2)!} \binom{r}{k-1} \frac{(m+r-k)!}{(m-k-1)!} \frac{(n+r-k)!}{(n-k-1)!} (k^2r - 3kr^2 - mn r + 2r^3 \\ & + k^2 + km + kn - 7kr - 3mn - mr - nr + 7r^2 - 4k - m - n + 8r + 3). \end{aligned}$$

This can be verified by substitution and cancellation.

$G(r, k)$  is zero when  $k < 1$  or  $k > r+1$ , so when we sum (16) over all  $k$ , the right hand side telescopes to zero, and each  $F(r, k)$  term in the left hand side sums to a  $P_r(m, n)$  term, so

$$(r+3)P_{r+2}(m, n) - (2r+3)(m+n)P_{r+1}(m, n) + r[(m-n)^2 - (r+1)^2]P_r(m, n) = 0,$$

which is equivalent by rearrangement to (15).  $\square$

Along with the initial cases  $P_1(m, n) = mn$  and  $P_2(m, n) = mn(m+n)$ , this gives us a simple method of computing any  $P_r(m, n)$  recursively.

(15) does not possess an obvious combinatorial interpretation, although it does make a number of properties of  $P_r(m, n)$  immediately clear, i.e. that  $P_r(m, n)$  is symmetric, and  $P_r(m, m)$  is a polynomial of order  $r+1$  which is even when  $r$  is odd and odd when  $r$  is even.

### 3.3 Beyond one face

The method used in Theorem 4 along with [5] can be used more generally to compute other generating functions which can be expressed in terms of the reduced density operator  $\hat{\rho}^A$ . This fact can be used to compute generating functions for counting rooted hypermaps with more than one face, and in this section we briefly outline how this can be done using the two-face case as an example.

**Theorem 6.** *Let*

$$P_{a,b}(m,n) = \frac{\Gamma(mn + a + b)}{\Gamma(mn)} \langle \text{Tr}[(\hat{\rho}^A)^a] \text{Tr}[(\hat{\rho}^A)^b] \rangle$$

for positive integers  $r$  and  $s$ , and let  $h_r^{(2)}(e,v)$  be the number of rooted hypermaps with two faces,  $e$  edges and  $v$  vertices. Then

$$\sum_{e,v} h_r^{(2)}(e,v) m^e n^v = \sum_{b=1}^{r-1} \frac{1}{b} (P_{r-b,b}(m,n) - P_{r-b}(m,n) P_b(m,n)).$$

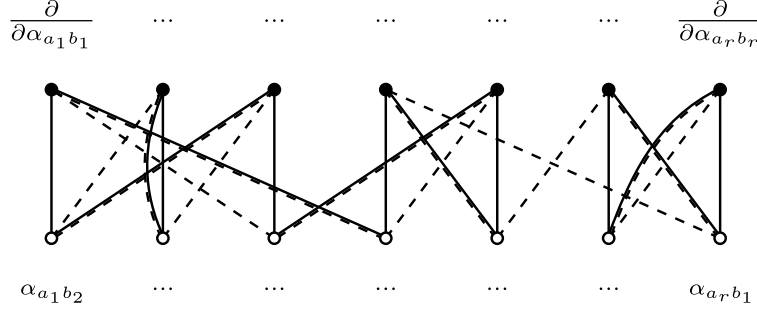


Figure 4: A diagram representing a hypermap with two face. This diagram shares most properties with figure 2, except that the dashed lines have been rearranged so that there are two loops of solid and dotted lines (of lengths three and four going from left to right). This diagram in particular has been chosen because it is connected; if the permutation associated with the double lines did not join the two loops together the diagram (and resulting hypergraph) would be disjoint, meaning it could not represent a hypermap.

*Proof.* Figure 2 demonstrates a representation of rooted one-face hypermaps as diagrams with two distinct sets of  $r$  vertices and three distinct sets of  $r$  edges connecting them. Two of these edge types – the solid and dashed edges – combine to create a single closed loop, which corresponds to the single face of the hypermap it represents.

We assign to a two-face rooted hypermap a similar representation, but with the solid and dashed edges forming two loops instead of one. An example of

such a diagram is shown in Figure 4. If we take a diagram of this form and sum over all possible arrangements of the double lines, then in the same way that diagrams with one loop of length  $r$  corresponds to

$$P_r(m, n) = \frac{\Gamma(mn + r)}{\Gamma(mn)} \langle \text{Tr}[(\hat{\rho}^A)^r] \rangle,$$

a diagram with two loops of length  $a$  and  $b$  corresponds to

$$P_{a,b}(m, n) = \frac{\Gamma(mn + a + b)}{\Gamma(mn)} \langle \text{Tr}[(\hat{\rho}_m^{mn})^a] \text{Tr}[(\hat{\rho}_m^{mn})^b] \rangle.$$

Proving this requires following the same method used in Theorem 1, and seeing that each trace term in the mean corresponds to a single loop.

This function is not sufficient to give us the generating function for two-face rooted hypermaps, however, as it over-counts for two reasons. First, it counts cases where the diagram is disjoint. We remove these cases by subtracting

$$P_a(m, n)P_b(m, n),$$

which generates the disjoint cases by treating the two loops as independent one-loop diagrams placed next to each other.

Secondly, as rooting the hypermaps requires only fixing of one of the two loops against cyclic permutation, there remains a cyclical degeneracy in the second loop; if the diagram isn't disjoint then cycling the second will necessarily produce a distinct diagram, so each diagram is in an equivalence class of size  $b$ . We account for this degeneracy by dividing by  $b$ .

Finally, to count all such hypermaps with  $r$  darts, we sum over all pairs of integers  $(a, b)$  which sum to  $r$ , giving the generating function

$$\sum_{e, v} h_r^{(2)}(e, v) m^e n^v = \sum_{b=1}^{r-1} \frac{1}{b} (P_{r-b, b}(m, n) - P_{r-b}(m, n) P_b(m, n)). \quad (17)$$

□

This generating function can then be evaluated exactly using the same tools we applied to the one-face case. But without going that far we can already derive the following result, a direct analogue of Corollary 1:

**Corollary 2.** *There are*

$$\sum_{b=1}^{r-1} \frac{r! - b!(r-b)!}{b}$$

*rooted hypermaps with two faces and  $r$  darts.*



*Proof.* (17) gives us a generating function for counting rooted hypermaps with two faces and  $r$  darts. In order to sum over all possible numbers of edges and vertices we simply set  $m$  and  $n$  to unity in this expression.

Due to the fact that  $\hat{\rho}^A$  has unit trace,

$$P_r(1,1) = \frac{\Gamma(1+r)}{\Gamma(1)} \langle 1 \rangle = r!$$

and

$$P_{a,b}(1,1) = (a+b)!,$$

so

$$\begin{aligned} \sum_{e,v} h_r^{(2)}(e,v) &= \sum_{b=1}^{r-1} \frac{P_{r-b}(1,1) - P_b(1)P_{r-b}(1)}{b} \\ &= \sum_{b=1}^{r-1} \frac{r! - b!(r-b)!}{b}. \end{aligned}$$

□

## 4 Discussion

We have seen that our method for finding closed-form generating functions has a quantitative advantage over the direct enumerative method in terms of the time it takes to compute the actual numbers of hypermaps. However, this is not the only benefit of the method.

Generating functions are a very powerful tool in enumerative combinatorics, as they allow numerous operations acting on the objects being counted to be represented by simple algebraic operations. For instance, the union of two sets is enumerated using the sum of their generating functions, and the product of two sets is enumerated by the product of their generating functions[17, pp 3-4].

The general methods used in this paper have potentially very wide applicability. All of the examples covered in this paper are possible because matrix polynomials integrated over the unit sphere have a connection to the cycle representation of permutations, as we showed in Sections 2.2 and 2.3. Many different combinatoric quantities can be expressed in terms of permutations (the 3-constellation representation of hypermaps provides just one example), so this method may be usable to construct generating functions in other cases where objects can be represented using permutations.

The fact that the matrix integrals used to compute the generating functions for one-face rooted hypermaps have a relation to bipartite quantum systems is also of interest. That there is a link between *bipartite* quantum systems and *bipartite* maps (equivalent to hypermaps) in particular is clearly not a coincidence; The parameters  $m$  and  $n$  in the function  $P_r(m,n)$  are identified with the dimensions of the two subsystems of the bipartite quantum system,

and are then also linked to the edges and vertices of the hypermap respectively (equivalently the two colours of vertices in the bipartite bicoloured map) when  $P_r$  is used as a generating function, so there is a clear correspondence between the bipartite natures of the quantum systems and the hypermaps.

The relation to quantum systems has had two immediate benefits here. First, the notation using means of functions of the reduced density matrix provides a useful shorthand for expressing the bulky matrix integrals and multiderivatives (see (3), for example), which is even extendable to multiple faces as seen in Section 3.3. Second, the use of expressions in terms of the reduced density matrix led directly to a method of evaluating the generating functions in closed form.

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## References

- [1] Didier Arquès. Hypercartes pointées sur le tore: Décompositions et dénombrements. *Journal of Combinatorial Theory, Series B*, 43(3):275–286, 1987.
- [2] D Bessis, C Itzykson, and J.B Zuber. Quantum field theory techniques in graphical enumeration. *Advances in Applied Mathematics*, 1(2):109–157, 1980.
- [3] E. Brézin, C. Itzykson, G. Parisi, and J. B. Zuber. Planar diagrams. *Communications in Mathematical Physics*, 59(1):35–51, 1978.
- [4] Louis Comtet. *Advanced Combinatorics: The Art of Finite and Infinite Expansions*. Springer, 1974.
- [5] Jacob P Dyer. Divergence of Lubkin's series for a quantum subsystem's mean entropy. *arXiv:1406.5776*, 2014.
- [6] Gerald B. Folland. How to Integrate a Polynomial over a Sphere. *The American Mathematical Monthly*, 108(5):446, 2001.
- [7] J. Harer and D. Zagier. The Euler characteristic of the moduli space of curves. *Inventiones Mathematicae*, 85(3):457–485, 1986.
- [8] B. R. Heap. Permutations by Interchanges. *The Computer Journal*, 6(3):293–298, 1963.

- [9] C. Itzykson and J.-B. Zuber. The planar approximation. II. *Journal of Mathematical Physics*, 21(3):411, 1980.
- [10] Sergei K Lando and Alexander K Zvonkin. *Graphs on surfaces and their applications*, volume 2. Springer, 2004.
- [11] Elihu Lubkin. Entropy of an  $n$ -system from its correlation with a  $k$ -reservoir. *Journal of Mathematical Physics*, 19(5):1028, 1978.
- [12] Alexander Mednykh and Roman Nedela. Enumeration of unrooted hypermaps of a given genus. *Discrete Mathematics*, 310(3):518–526, 2010.
- [13] Don Page. Average entropy of a subsystem. *Physical Review Letters*, 71(9):1291–1294, 1993.
- [14] M Petkovšek, HS Wilf, and D Zeilberger. *A = B*. A K Peters Ltd, 1996.
- [15] M. C. Ring. MAPM, A Portable Arbitrary Precision Math Library in C, 2001.
- [16] J. J. Sakurai and Jim Napolitano. *Modern Quantum Mechanics: Pearson New International Edition*. Pearson Education, Limited, 2013.
- [17] Richard P. Stanley. *Enumerative Combinatorics, Volume 1*. Cambridge University Press, 1997.
- [18] T. R. Walsh. Generating nonisomorphic maps and hypermaps without storing them, 2012.
- [19] T.R.S Walsh. Hypermaps versus bipartite maps. *Journal of Combinatorial Theory, Series B*, 18(2):155–163, 1975.
- [20] D Zeilberger. Maple Programs Related to  $A=B$ .
- [21] A. Zvonkin. Matrix integrals and map enumeration: An accessible introduction. *Mathematical and Computer Modelling*, 26(8-10):281–304, 1997.